Estimates of analysis error, forecast error, and predictability using the adjoint of 4D-Var

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Outline

- Analysis error covariance
- TL and AD of 4D-Var
- Functions of the state vector
- Examples for the California Current
- Forecast errors (and predictability)

Expected Analysis Error

Analysis: $$\mathbf{x_a} = \mathbf{x_b} + \mathbf{Kd}$$

Innovations: $$\mathbf{d} = \left(\mathbf{y} - H(\mathbf{x_b})\right)$$

Gain: $$\mathbf{K} = \mathbf{BH}^{\mathrm{T}}\left(\mathbf{HBH}^{\mathrm{T}} + \mathbf{R}\right)^{-1}$$

Analysis error covariance:

$$\mathbf{A} = \left(\mathbf{I} - \mathbf{KH}\right)\mathbf{B}\left(\mathbf{I} - \mathbf{KH}\right)^{\mathrm{T}} + \mathbf{KRK}^{\mathrm{T}}$$

In general: $$\mathbf{K} \rightarrow \tilde{\mathbf{K}}$$ (the "practical gain" matrix)

$$\tilde{\mathbf{A}} = \left(\mathbf{I} - \tilde{\mathbf{K}}\mathbf{H}\right)\mathbf{B}\left(\mathbf{I} - \tilde{\mathbf{K}}\mathbf{H}\right)^{\mathrm{T}} + \tilde{\mathbf{K}}\mathbf{R}\tilde{\mathbf{K}}^{\mathrm{T}}$$

Practical Estimates of A

- **Reduced rank approx of A (e.g. Fisher & Courtier, 1995)**
  - over estimates expected errors

- **Ensemble Kalman Filters**
  - localization and inflation issues

- **Ensemble 4D-Var (e.g. Belo Pereira & Berre, 2006)**
  - ensemble covariance mimics A
  - computationally demanding

TL and AD of 4D-Var

Analysis: $\mathbf{x_a} = \mathbf{x_b} + \mathcal{K}\left(\mathbf{d}, p_a, p_m\right)$

Inspired by ensemble 4D-Var:

$$\delta\mathbf{x_a} \simeq \left(\partial\mathcal{K}/\partial\mathbf{d}\right)\delta\mathbf{d}$$

← δy with covariance R

δx_b with covariance B

Expected analysis error covariance:

$$\hat{\mathbf{A}} = \left\langle \delta\mathbf{x_a}\delta\mathbf{x_a}^{\mathrm{T}} \right\rangle$$

$$\hat{\mathbf{A}} = \left(\mathbf{I} - \left(\partial\mathcal{K}/\partial\mathbf{d}\right)\mathbf{H}\right)\mathbf{B}\left(\mathbf{I} - \left(\partial\mathcal{K}/\partial\mathbf{d}\right)\mathbf{H}\right)^{\mathrm{T}}$$

$$+\left(\partial\mathcal{K}/\partial\mathbf{d}\right)\mathbf{R}\left(\partial\mathcal{K}/\partial\mathbf{d}\right)^{\mathrm{T}}$$

At limit of convergence: $\left(\partial\mathcal{K}/\partial\mathbf{d}\right) = \mathbf{K}$

Expected Analysis Errors of Functions

Linear scalar functions: $\mathcal{J}(\mathbf{x}) = \mathbf{w}^{\mathrm{T}}\mathbf{x}$

(e.g. transport, heat content)

Quadratic scalar functions: $\mathcal{J}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{L}\mathbf{x}$

(e.g. eddy kinetic energy)

For linear functions:

***Prior* error variance** $\left(\sigma_{\mathcal{J}}^{\mathbf{b}}\right)^2 = \mathbf{w}^{\mathrm{T}}\mathbf{B}\mathbf{w}$

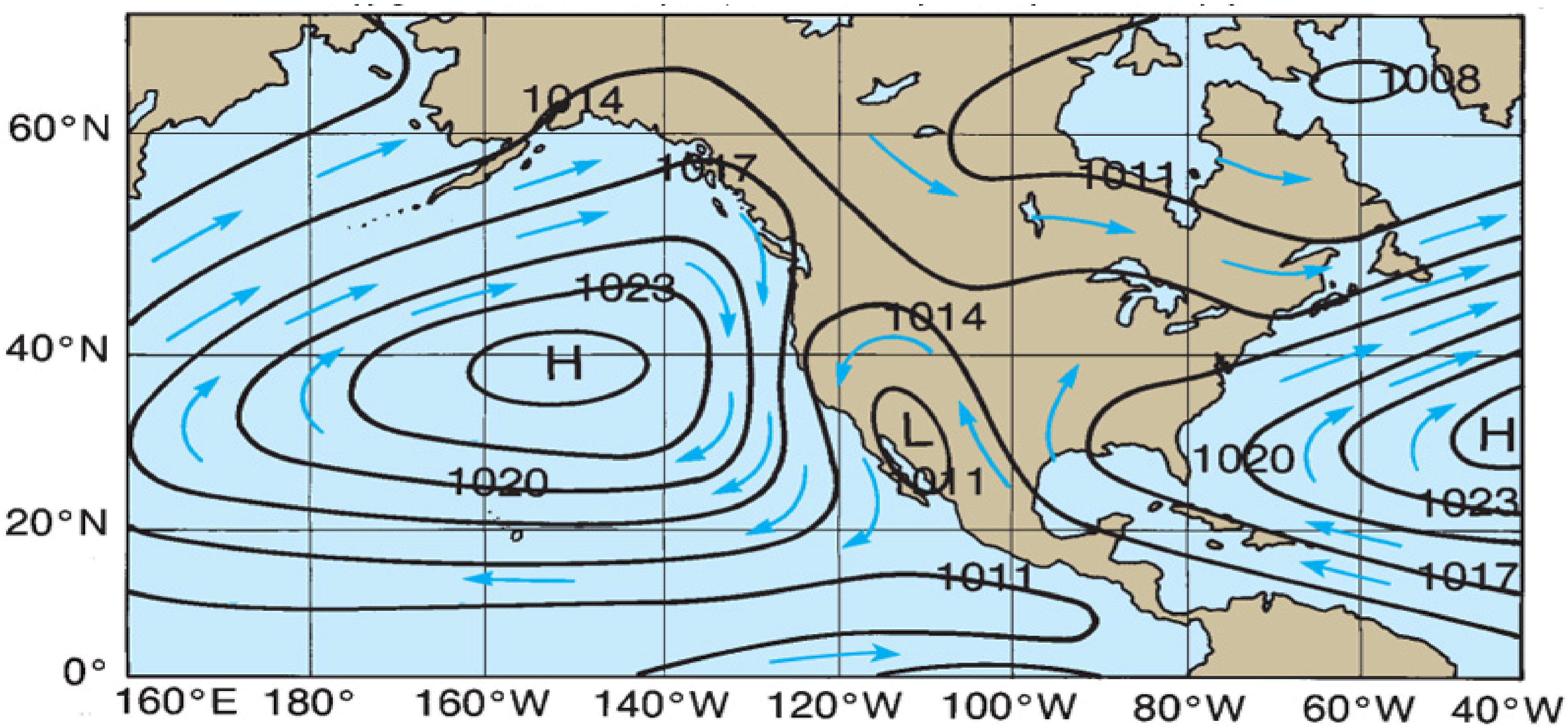
***Posterior* error variance** $\left(\sigma_{\mathcal{J}}^{\mathbf{a}}\right)^2 = \mathbf{w}^{\mathrm{T}}\mathbf{A}\mathbf{w}$

$\mathbf{A}$ **is given by** $\hat{\mathbf{A}}$ **or** $\tilde{\mathbf{A}}$

**No Ensembles
Were Harmed in these
Calculations**

**Results pertain to an infinite
theoretical ensemble**

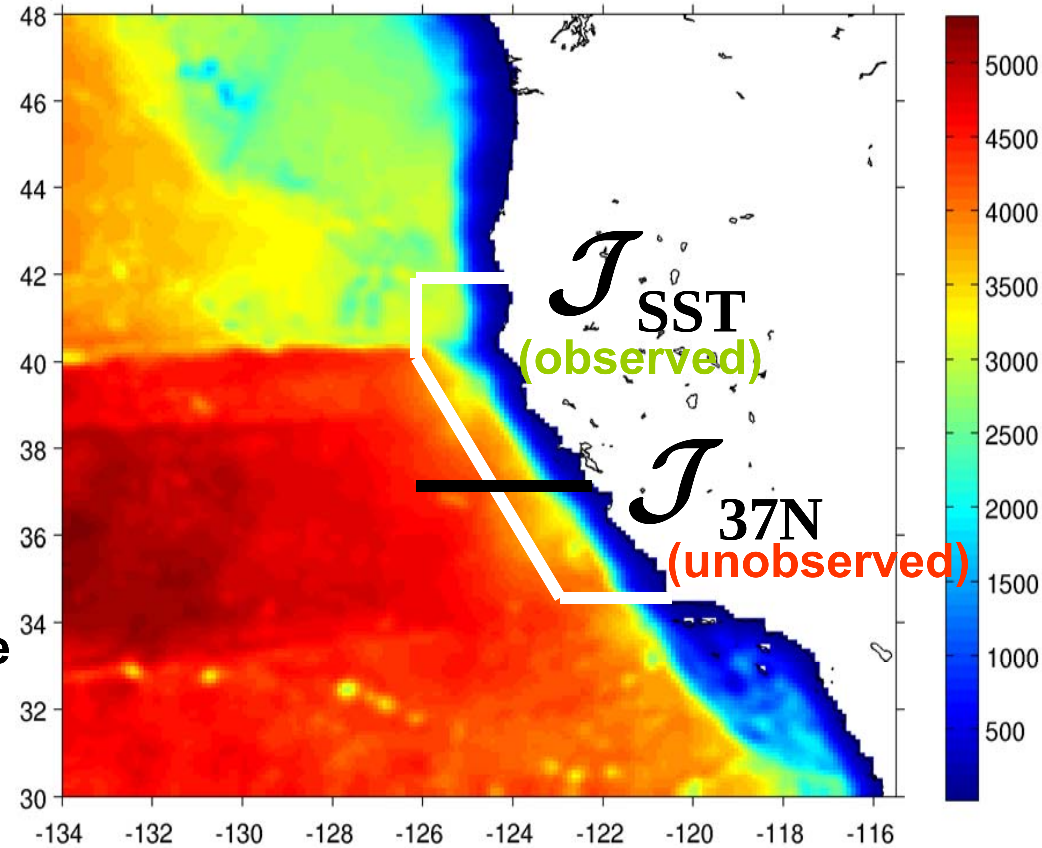
The California Current



ROMS: California Current System (CCS)

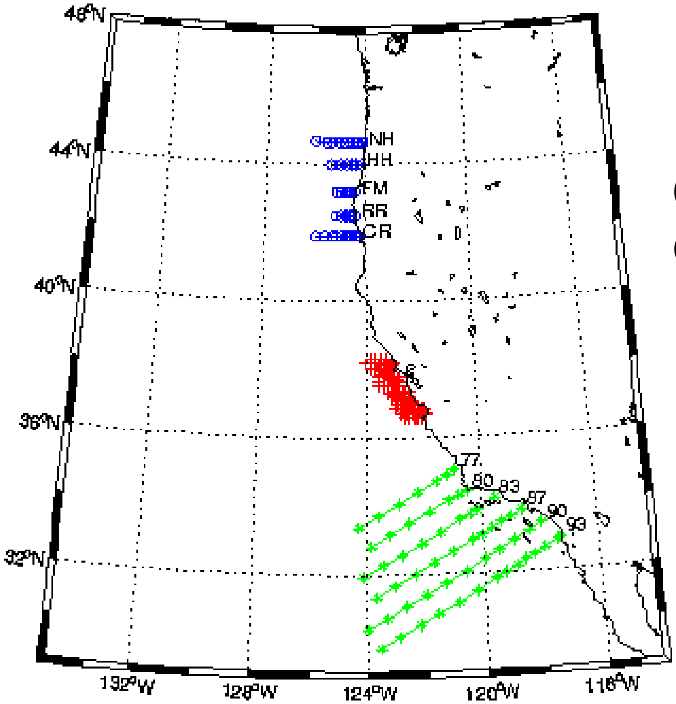
(Regional Ocean Modeling System)

- ROMS: PE, hydrostatic, sigma
- 30 km, 30 levels
- 4D-Var: Dual, 1 outer, 60 inner
- Minimum residual descent
- Control vector: **x**(0), **f**(t), **b**(t)
- COAMPS forcing, $f_b(t)$
- ECCO open b.c.s, $b_b(t)$
- *Prior* $x_b(0)$ from previous cycle
- 7 day assimilation cycles
- July 2002 – Dec. 2004

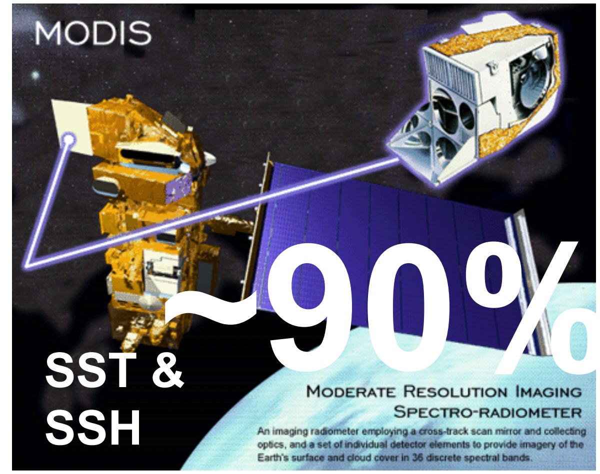


Veneziani et al (2009)
Broquet et al (2009a,b, 2011)
Moore et al (2011a,b,c)

Observations (y)

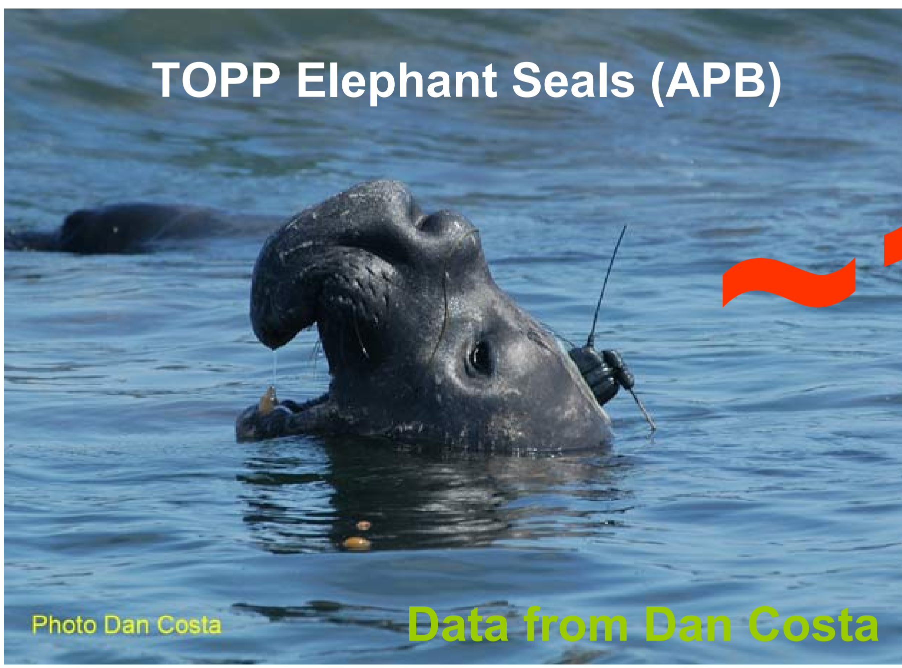


CalCOFI & GLOBEC



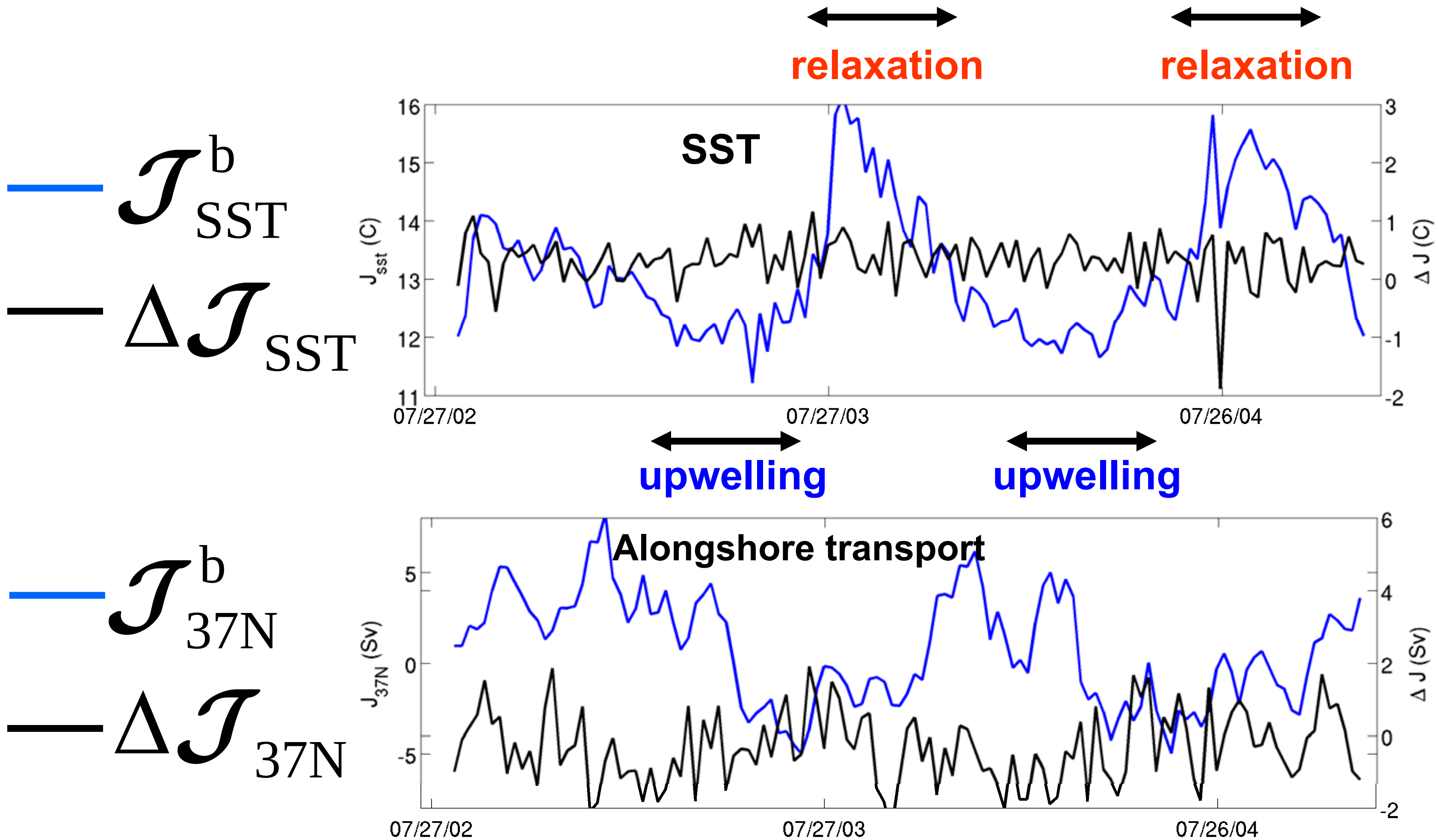
EN3

Ingleby and Huddleston (2007)



~10%

Time Series of $\mathcal{J}_{SST}$ and $\mathcal{J}_{37N}$

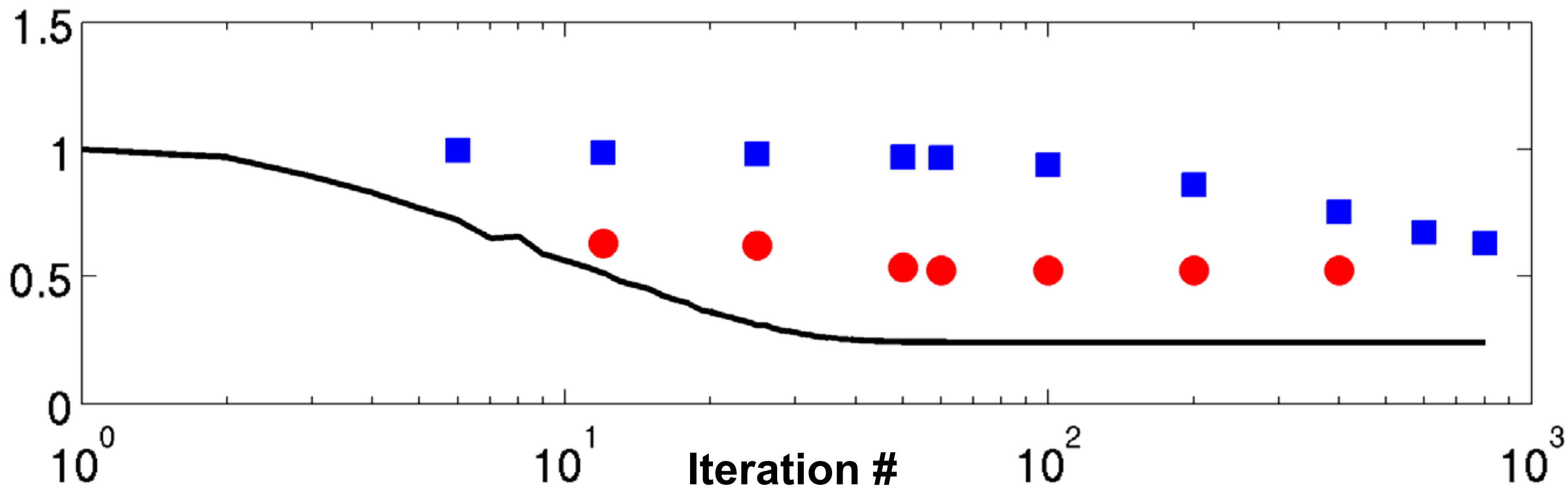


Convergence Properties of $\tilde{\mathbf{A}}$ and $\hat{\mathbf{A}}$

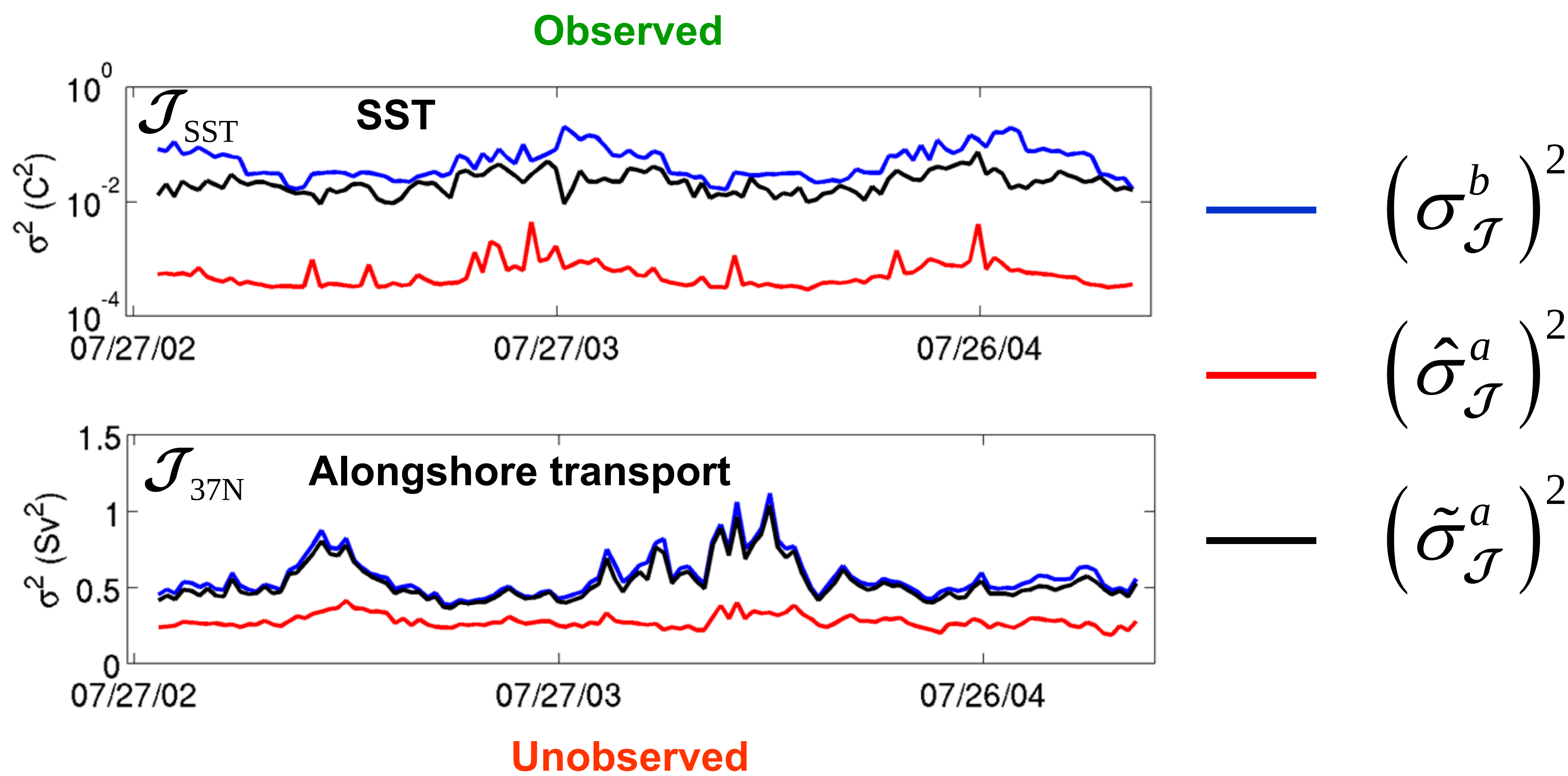
$$\tilde{\mathbf{A}} = \left(\mathbf{I} - \tilde{\mathbf{K}}\mathbf{H}\right)\mathbf{B}\left(\mathbf{I} - \tilde{\mathbf{K}}\mathbf{H}\right)^{\mathrm{T}} + \tilde{\mathbf{K}}\mathbf{R}\tilde{\mathbf{K}}^{\mathrm{T}}$$

$$\hat{\mathbf{A}} = \left(\mathbf{I} - \left(\frac{\partial \mathcal{K}}{\partial \mathbf{d}}\right)\mathbf{H}\right)\mathbf{B}\left(\mathbf{I} - \left(\frac{\partial \mathcal{K}}{\partial \mathbf{d}}\right)\mathbf{H}\right)^{\mathrm{T}} + \left(\frac{\partial \mathcal{K}}{\partial \mathbf{d}}\right)\mathbf{R}\left(\frac{\partial \mathcal{K}}{\partial \mathbf{d}}\right)^{\mathrm{T}}$$

■ $\tilde{\sigma}_a^2 / \sigma_b^2$ ● $\hat{\sigma}_a^2 / \sigma_b^2$ — J



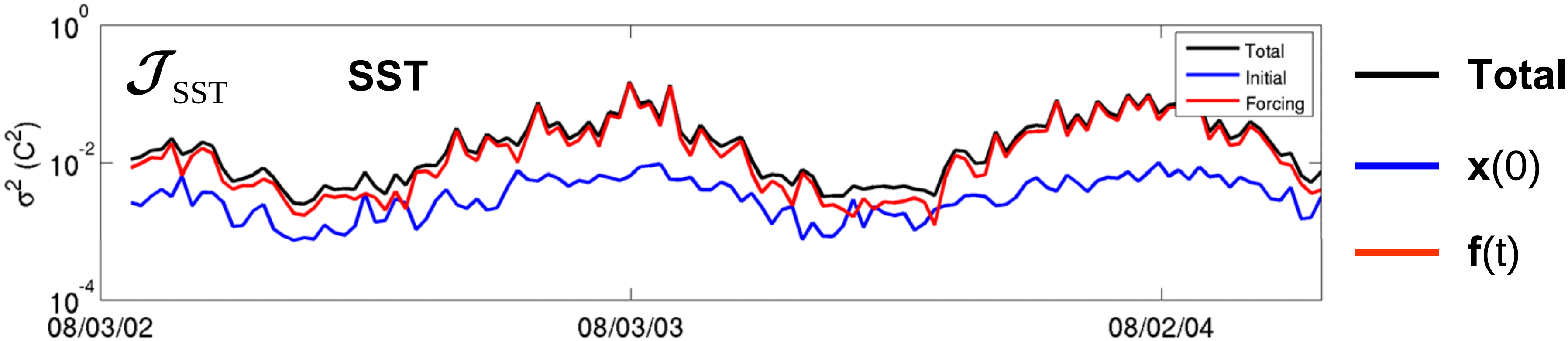
Prior and *Posterior* Errors

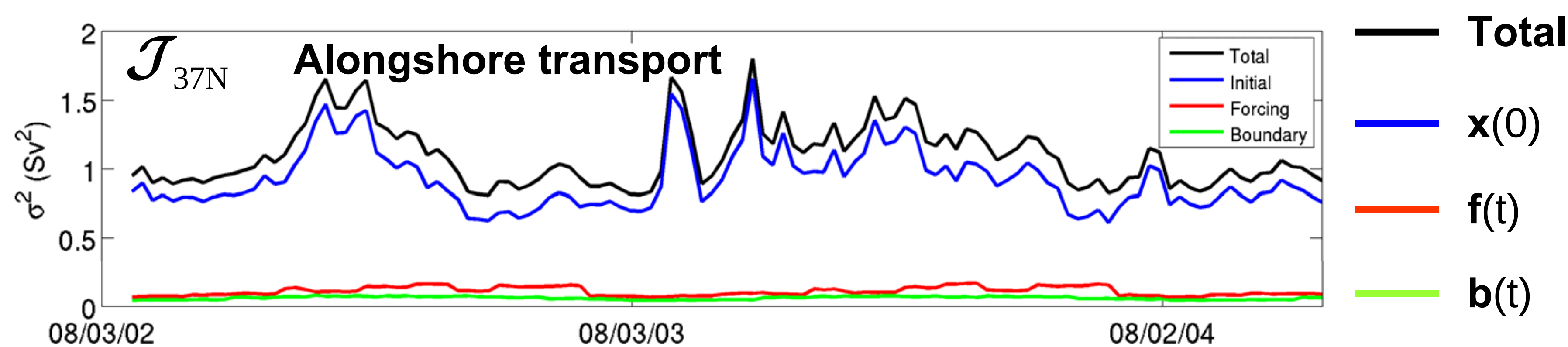


Expected Forecast Errors

$$\mathbf{P^f} \simeq \mathbf{M_f D M_f^T} \qquad \mathbf{D} = \mathrm{diag}\left(\mathbf{M_a \hat{A} M_a^T}, \mathbf{B_f}, \mathbf{B_b}\right)$$

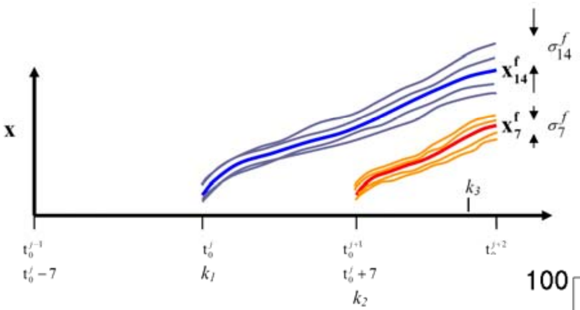
$$\left(\sigma_{\mathcal{J}}^{\mathbf{f}}\right)^2 = \mathbf{w^T P^f w}$$

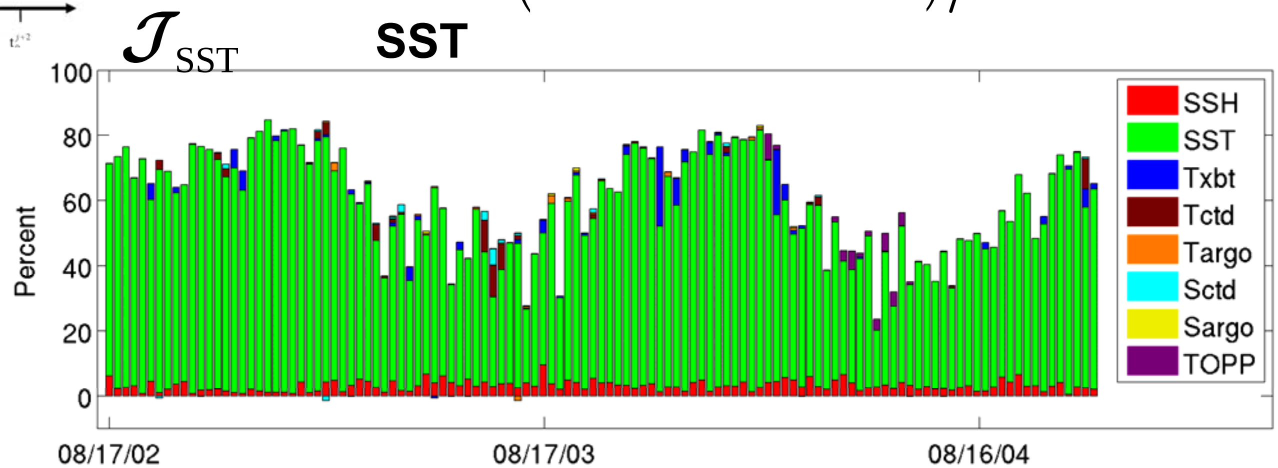




Predictability

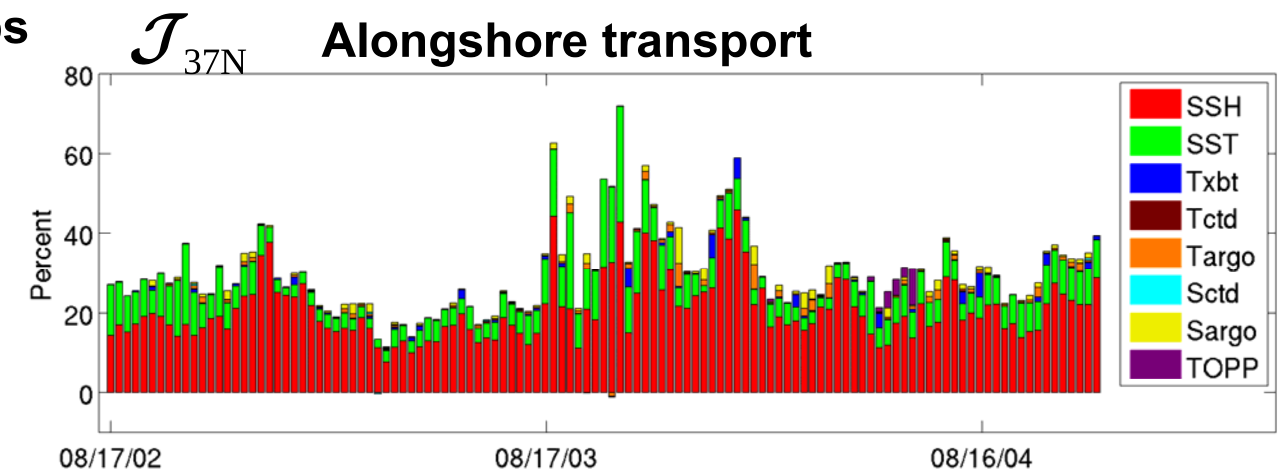
$$r = 100\left(\left(\sigma_{14}^{f}\right)^{2} - \left(\sigma_{7}^{f}\right)^{2}\right) \Big/ \left(\sigma_{14}^{f}\right)^{2}$$





$$r > 0$$

positive impact of obs on predictability



Analogous to obs impact (Langland & Baker; Errico; Gelaro et al)

Final Remarks

- **One integration of $\left(\partial\mathcal{K}/\partial\mathbf{d}\right)^{\mathrm{T}}$ per function**

- **Consistency checks:**

$$\boxed{\mathcal{J}^{\mathbf{a}} - \mathcal{J}^{\mathbf{b}} \sim N\left(0,\left(\sigma_b^2 - \sigma_a^2\right)^{1/2}\right)}$$

$\pm 2\left(\sigma_b^2 - \sigma_a^2\right)^{1/2}$ **(69-82%)**

$$\boxed{\mathcal{J}^{\mathbf{f}} - \mathcal{J}^{\mathbf{a}} \sim N\left(0,\sigma^f\right)}$$

$\pm 2\sigma^f$ **46% ($\mathcal{J}_{\mathrm{SST}}$) 91% ($\mathcal{J}_{\mathrm{37N}}$)**

→ reliability

- **Extension to quadratic functions**

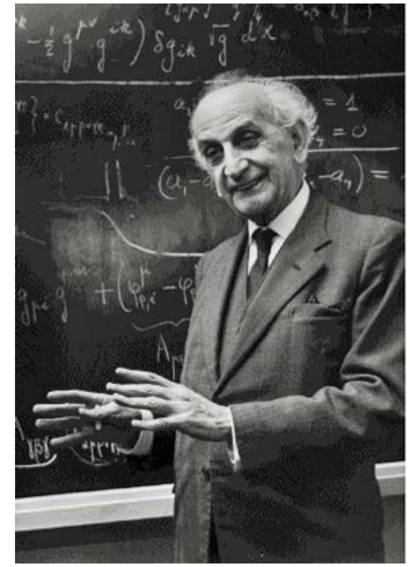
- $\left(\partial\mathcal{K}/\partial\mathbf{d}\right)^{\mathrm{T}}$ **can predict change in error covariance due to withhold obs → optimal array design**

Lanczos Factorization

$$\tilde{\mathbf{K}} \approx \mathbf{B}\mathbf{H}^{\mathrm{T}}\mathbf{V}_{\mathrm{m}}\mathbf{W}_{\mathrm{m}}\mathbf{V}_{\mathrm{m}}^{\mathrm{T}}$$

Dual form
$m \ll N_{obs}$

V_m is the matrix of orthonormal Lanczos vectors
m is the number of inner-loops



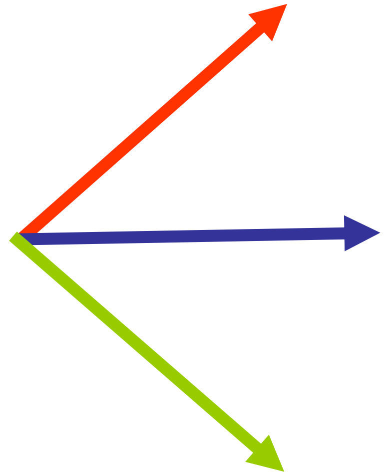
Cornelius Lanczos (1893-1974)

$$\delta\mathbf{x}_{\mathbf{a}} \simeq \left(\partial\mathcal{K}/\partial\mathbf{d}\right)\Delta\mathbf{d} \simeq \mathbf{B}\mathbf{H}^{\mathrm{T}}\mathbf{V}_{\mathrm{m}}\mathbf{W}_{\mathrm{m}}\mathbf{V}_{\mathrm{m}}^{\mathrm{T}}\Delta\mathbf{d} + \mathbf{B}\mathbf{H}^{\mathrm{T}}[\mathbf{V}_{\mathrm{m}}\mathbf{W}_{\mathrm{m}}\boxed{\Delta\mathbf{V}_{\mathrm{m}}^{\mathrm{T}}}\ldots + \mathbf{V}_{\mathrm{m}}\Delta\mathbf{W}_{\mathrm{m}}\mathbf{V}_{\mathrm{m}}^{\mathrm{T}}\ldots + \boxed{\Delta\mathbf{V}_{\mathrm{m}}}\mathbf{W}_{\mathrm{m}}\mathbf{V}_{\mathrm{m}}^{\mathrm{T}}]\mathbf{d}$$

The perturbed Lanczos vectors $\Delta\mathbf{V}_{\mathrm{m}}$ span obs space via $\langle\bullet\rangle$

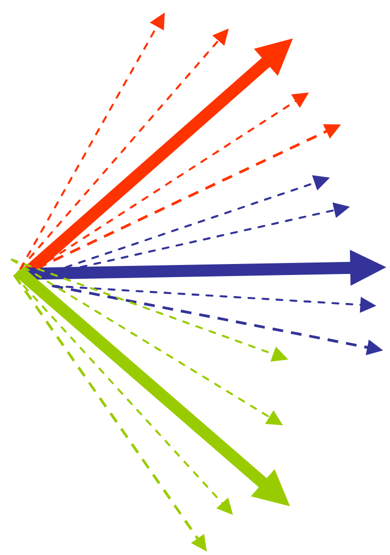
Perturbation Lanczos Vectors

$$\tilde{\mathbf{K}}:$$



m orthonormal Lanczos vectors
in obs space where
$m << N_{obs}$

$$\left(\frac{\partial \mathcal{K}}{\partial \mathbf{d}}\right)^{\mathrm{T}}:$$



Perturbed Lanczos vectors
(via <>) more effectively span
obs space