Model-reduced 4D-Var data assimilation in application to 1D ecosystem model

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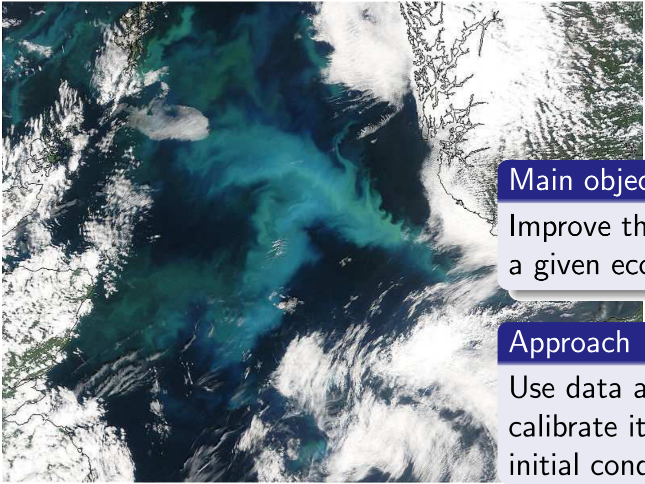
Outline

Introduction



Main objective

Improve the predictions of a given ecological model

Approach

Use data assimilation to calibrate its parameters & initial conditions

1D Ecological Model

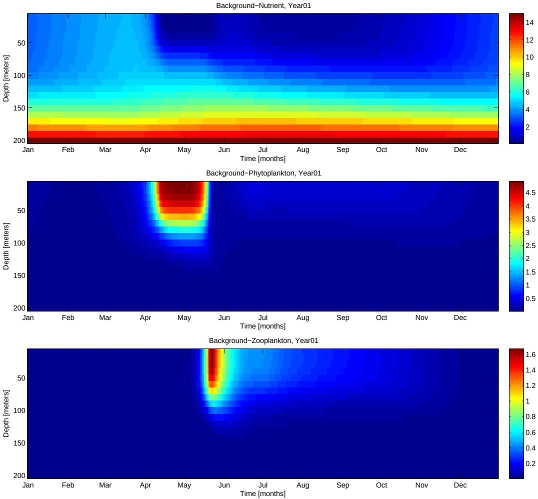
State variables

1. Nutrients (N)
2. Phytoplankton (P)
3. Herbivorous zooplankton (H)

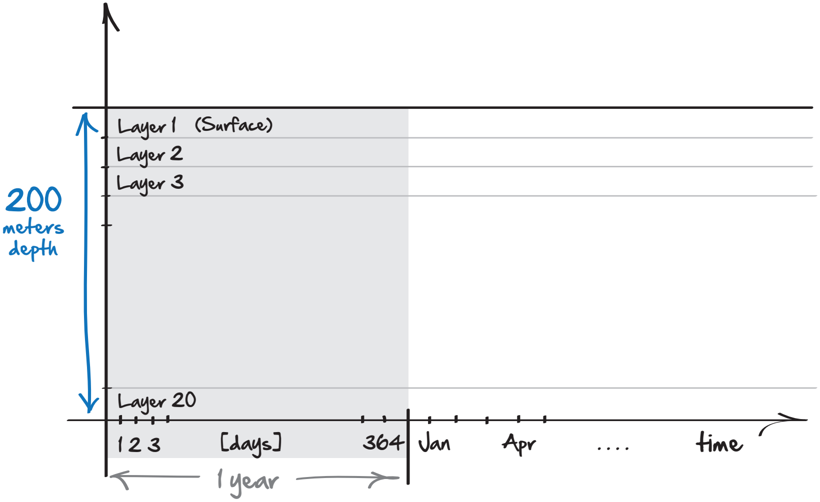
Estimated parameters

f grazing efficiency
g loss to carnivores
r plant metabolic loss

Evans and Parslow (1985)
Eknes and Evensen (2002)



1D Ecological Model



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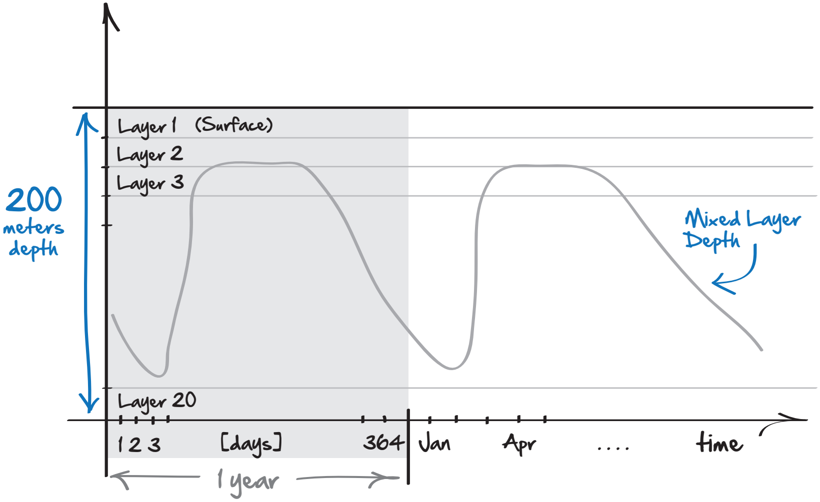

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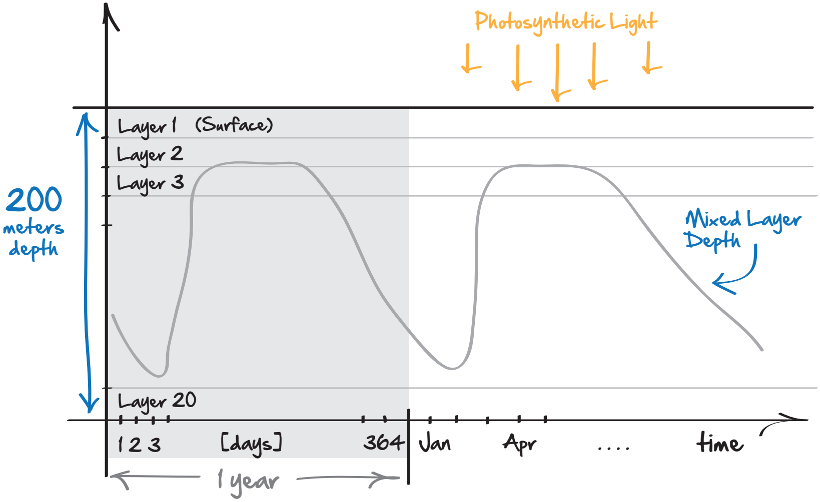

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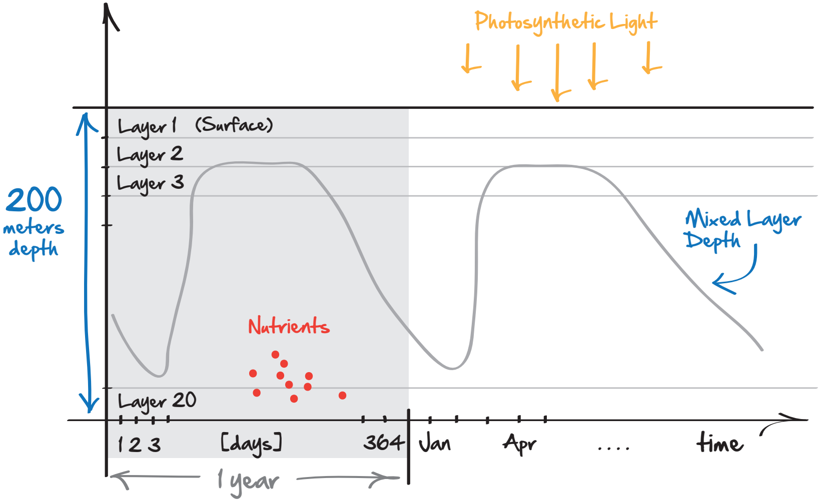

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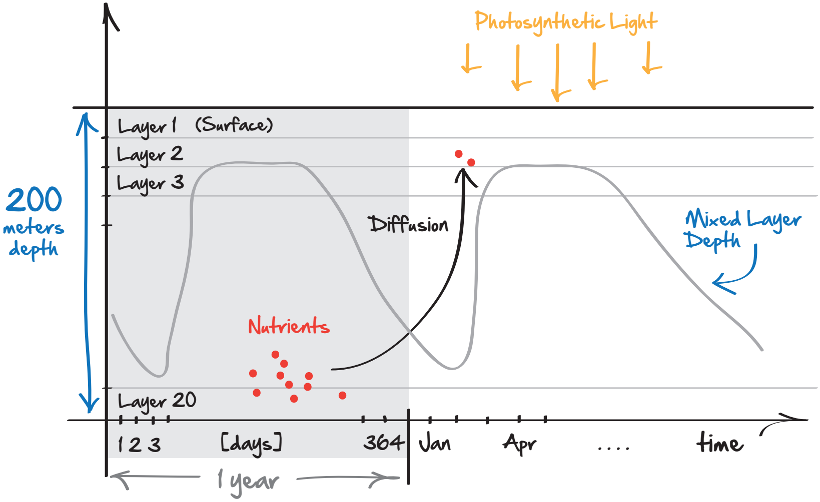

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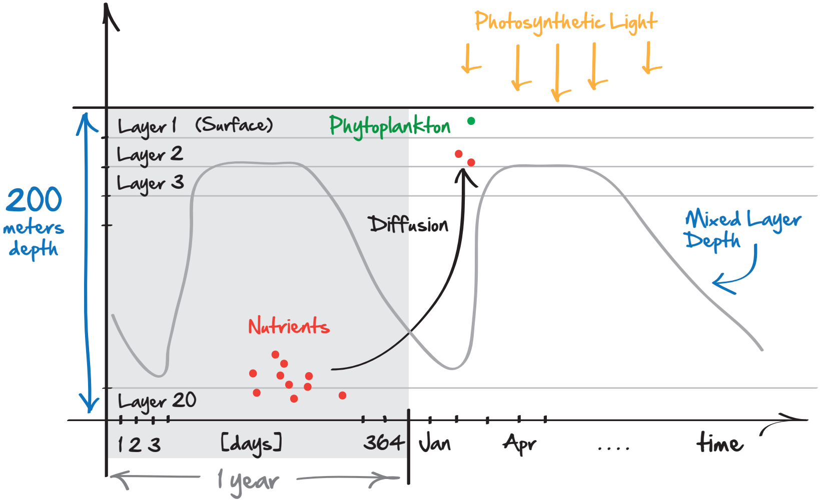

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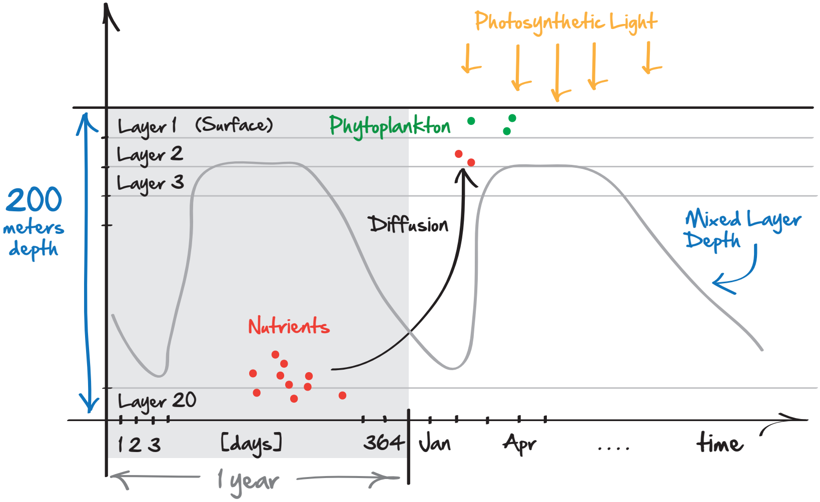

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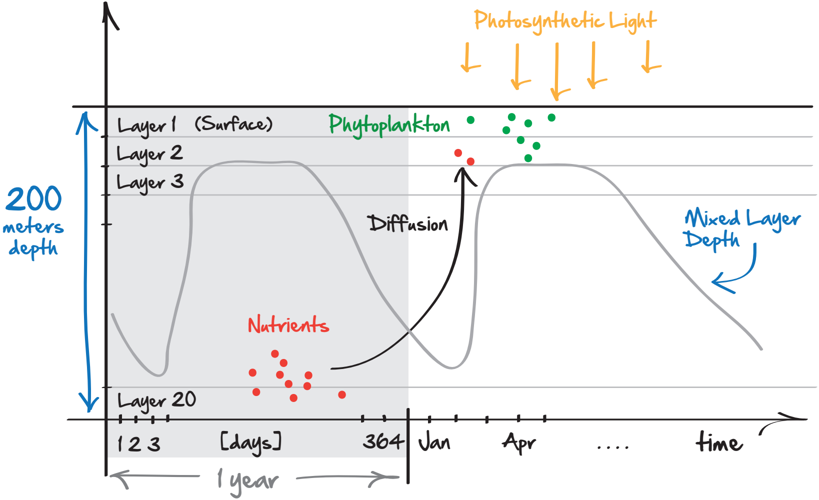

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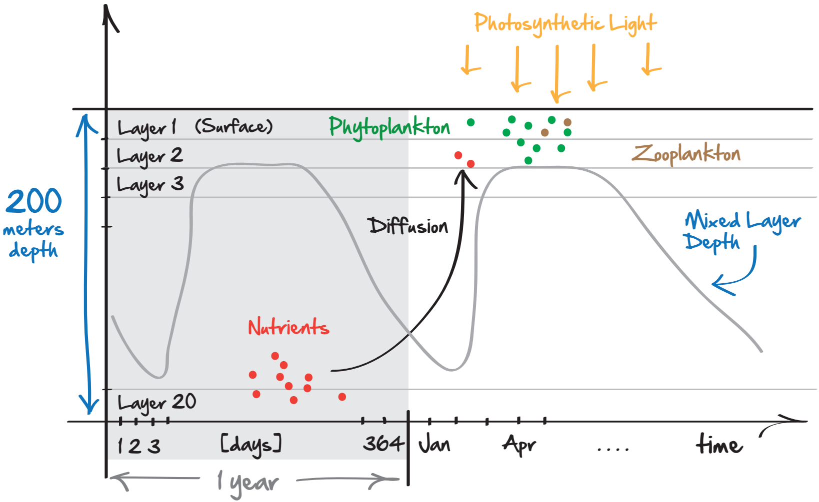

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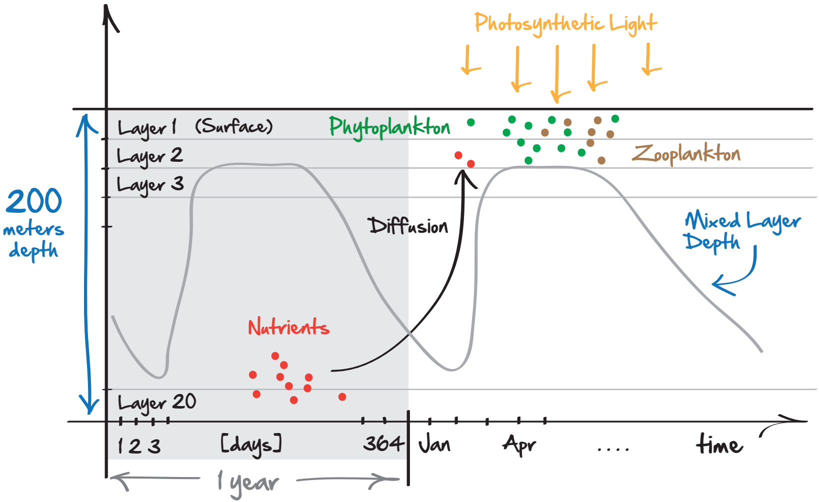

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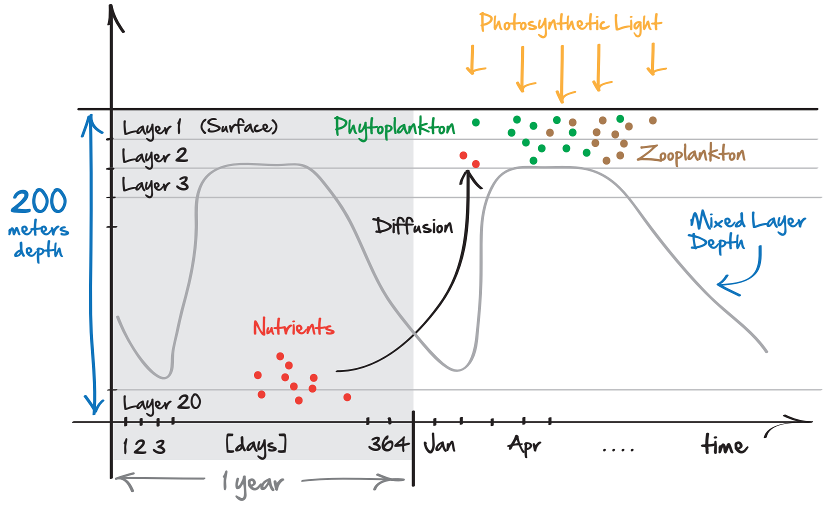

4D-Var: 4D Variational Data Assimilation

Cost function in 4D-Var

$$J(\alpha) = \sum_{i=1}^{n} \underbrace{(\mathcal{H}_i(x_i) - y_i)^T}_{Distances} R_i^{-1} \underbrace{(\mathcal{H}_i(x_i) - y_i)}_{Distances} + \underbrace{(\alpha - \alpha_b)^T B^{-1} (\alpha - \alpha_b)}_{Background \quad term}$$

minimization with constraints:
$x_i = \mathcal{M}_i(x_{i-1}, \alpha)$

B background error covariance matrix
R_i observation error covariance matrix

To minimize J over α, we need: $\nabla_\alpha J(\alpha)$

Needed to get $\nabla_\alpha J(\alpha)$

- exact derivatives of the model
- approximate the derivatives of the model with finite differences

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- **Complicated**
- approximate the derivatives of the model with finite differences

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Needed to get $\nabla_\alpha J(\alpha)$

- **Complicated**
- **Time consuming**

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To minimize J over α, we need:
$\nabla_\alpha J(\alpha)$

Solution

POD Model-reduced 4D-Var
(Vermeulen and Heemink, MWR, 2006)

POD Model Reduced 4D-Var

Incremental cost function in 4D-Var

$$J(\delta\alpha) = \sum_{i=1}^{n} (\mathbf{H}_i(\delta x_i, \delta\alpha) + d_i)^T R_i^{-1} (\mathbf{H}_i(\delta x_i, \delta\alpha) + d_i) + \delta\alpha^T B^{-1} \delta\alpha$$

minimization with constraints

$$\delta x_i = \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial x_{i-1}} \delta x_{i-1} + \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial \alpha} \delta\alpha$$

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With a small number of parameters, the finite differences method is feasible

POD Model Reduced 4D-Var

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minimization with constraints

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If the size of the state x is huge, the finite differences would be too expensive

POD Model Reduced 4D-Var

Incremental equation

$$\delta x_i = \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial x_{i-1}} \delta x_{i-1} + \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial \alpha} \delta \alpha$$

Project the increments into smaller subspace

$$P^T \delta x_i = P^T \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial x_{i-1}} \delta x_{i-1} + P^T \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial \alpha} \delta \alpha$$

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$$P^T \delta x_i = P^T \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial x_{i-1}} P\, P^T \delta x_{i-1} + P^T \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial \alpha} \delta \alpha$$

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$$\underbrace{P^T \delta x_i}_{\delta z_i} = P^T \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial x_{i-1}} P \underbrace{P^T \delta x_{i-1}}_{\delta z_{i-1}} + P^T \frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial \alpha} \delta \alpha$$

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$\delta z_i = P^T \delta x_i$ increment of the state in the reduced space

POD Model Reduced 4D-Var

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$\delta z_i = P^T \delta x_i$ increment of the state in the reduced space

$$\frac{\partial \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\partial x_{i-1}} p \simeq \frac{\mathcal{M}_i(x_{i-1}^b + \epsilon p, \alpha^b) - \mathcal{M}_i(x_{i-1}^b, \alpha^b)}{\epsilon}$$

directional derivative approximation

How to get matrix P?

We want to project into a smaller subspace, such that the most important dynamics of the system are kept

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STEP 1: Generate ensemble of perturbed model simulations

$$\alpha^b + \Delta\alpha_1 \rightarrow x_1^{\Delta 1}, \quad x_2^{\Delta 1}, \quad \dots \quad x_n^{\Delta 1}$$
$$\alpha^b + \Delta\alpha_2 \rightarrow x_1^{\Delta 2}, \quad x_2^{\Delta 2}, \quad \dots \quad x_n^{\Delta 2}$$

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Step 2: Create a covariance matrix

$$\mathbf{C_X} = \mathbf{\Delta X}\,\mathbf{\Delta X}^{\mathbf{T}}/(\mathbf{n}-\mathbf{1})$$

How to get matrix P?

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STEP 2: Create a covariance matrix

$$\mathbf{C_X} = \mathbf{\Delta X}\ \mathbf{\Delta X}^{\mathbf{T}} / (\mathbf{n} - \mathbf{1})$$

STEP 3: Decompose $\mathbf{C_X}$ with eigenvalue decomposition

$$\mathbf{\Delta X}\ \mathbf{\Delta X}^{\mathbf{T}} / (\mathbf{n} - \mathbf{1}) = \mathbf{P D P}^{\mathbf{T}}$$

P - eigenvectors, **D** - diagonal matrix with eigenvalues

How to get matrix P?

Step 1: Generate ensemble of perturbed model simulations

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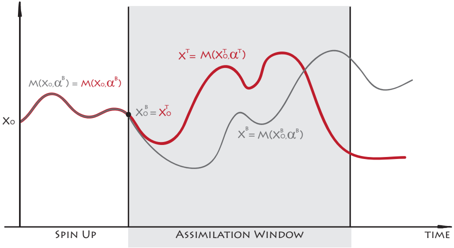
$$\mathbf{P}^{\mathbf{T}}\mathbf{\Delta X}\ (\mathbf{P}^{\mathbf{T}}\mathbf{\Delta X})^{\mathbf{T}}/(\mathbf{n}-\mathbf{1}) = \mathbf{\Delta Z}\ \mathbf{\Delta Z}^{\mathbf{T}}/(\mathbf{n}-\mathbf{1}) = \mathbf{D}$$

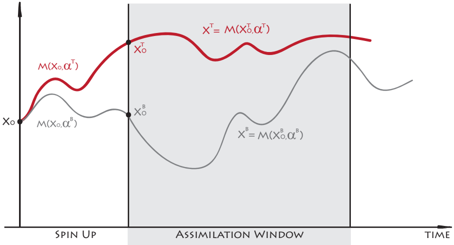
Twin Experiment Setup



Perfect Initial Condition

The spin up starts with

- prior par. for prior sol.
- prior par. for true sol.



Perturbed Initial Condition

The spin up starts with

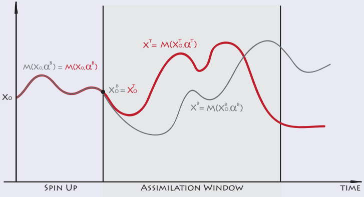
- prior par. for prior sol.
- true par. for true sol.

Experiment 1 - Estimate: Parameters

Experiment Setup - **Estimate Parameters**

Perfect Initial Condition

Spin up time	6 years
Assimilation time	4 years



Parameters Setup

	f	**g**	**r**
Prior	0.50	0.07	0.07
Truth	**0.90**	**0.11**	**0.11**
	80%	57%	57 %

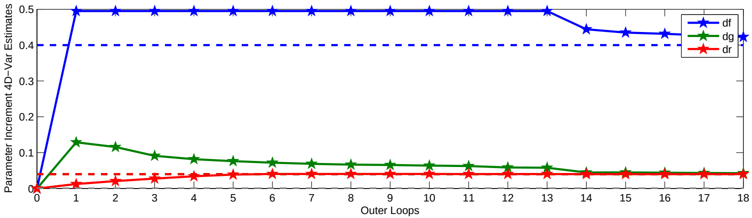
Observation Setup

Observe

What	surface Phyto
When	every 4 days
Error	30 %

Experiment 1 - Estimate: Parameters

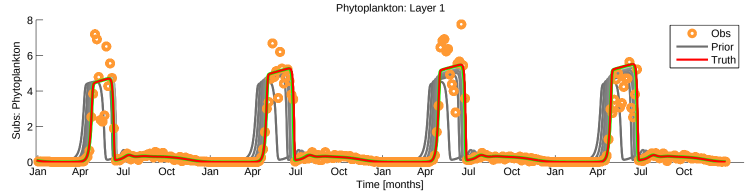
	Parameter Estimation			Cost Function	
	f	**g**	**r**		
Prior	0.5	0.07	0.07	Prior	3.91e+6
Truth	0.9	0.11	0.11	$\frac{1}{2}$Obs	364
4DVar	**0.9223**	**0.1127**	**0.1099**	4DVar	**340.45**

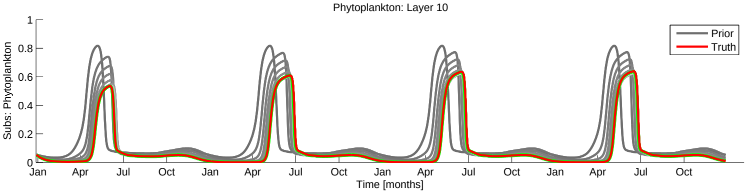


Parameter Increments shown within outer loops

Experiment 1 - Estimate: Parameters

Phytoplankton within time shown at the surface layer and at the 10th layer of the water column.





Grey - Outer Loops, **Green - 4DVar**

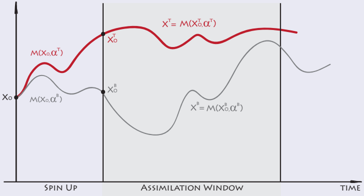
Red - Truth, **Orange - Observations**

Experiment 2 - Parameters & Initial Condition

Experiment Setup - **Estimate Parameters & Initial Condition**

Perturbed Initial Condition

Spin up time	455 days
Assimilation time	5 years



Parameters Setup

	f	g	r
Prior	0.50	0.07	0.07
Truth	**0.90**	**0.11**	**0.11**
	80%	57%	57 %

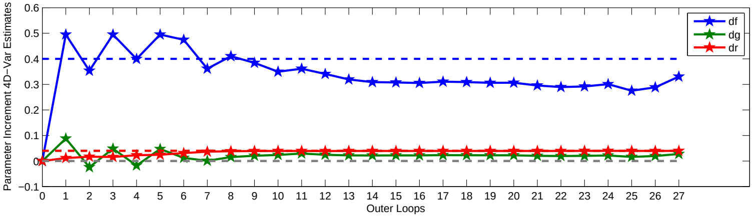
Observation Setup

Observe

What	surface Phyto
When	every 4 days
Error	30 %

Experiment 2 - Parameters & Initial Condition

	Parameter Estimation			Cost Function	
	f	**g**	**r**		
Prior	0.5	0.07	0.07	Prior	5.49e+6
Truth	0.9	0.11	0.11	$\frac{1}{2}$Obs	433
4DVar	**0.8304**	**0.0976**	**0.1098**	4DVar	**611.02**



Parameter Increments shown within outer loops

Experiment 2 - Parameters & Initial Condition

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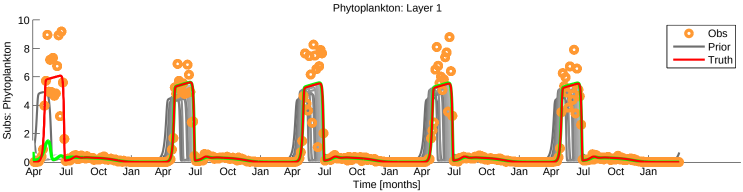
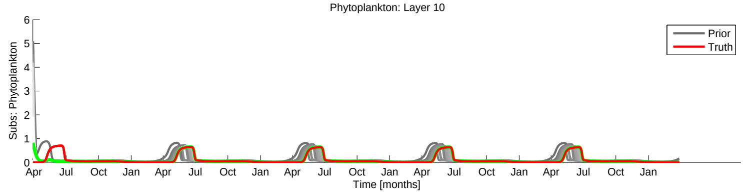


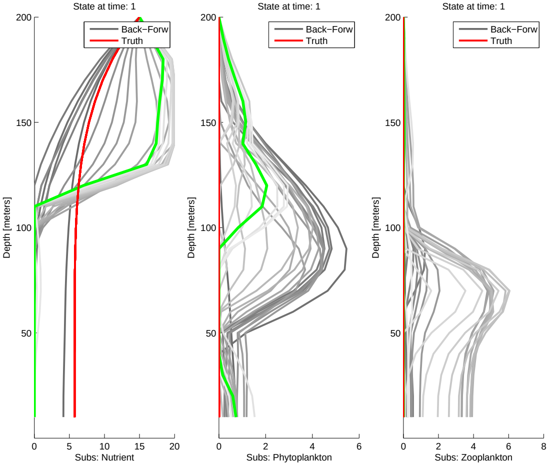


Grey - Outer Loops, **Green - 4DVar**

Red - Truth, **Orange - Observations**

Experiment 2 - Parameters & Initial Condition

Initial Condition estimation

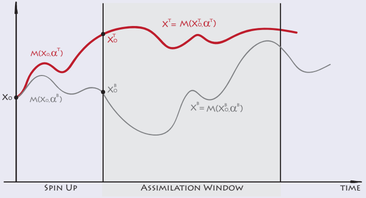


Grey - Outer Loops
Green - 4DVar
Red - Truth

Experiment 3 - Unrealistic Setup

Experiment Setup - **Estimate Parameters & Initial Condition**

Perturbed Initial Condition	
Spin up time	730 days
Assimilation time	5 years



Parameters Setup

	f	**g**	**r**
Prior	0.50	0.07	0.07
Truth	**0.5444**	**0.0766**	**0.0766**
	9%	9%	9 %

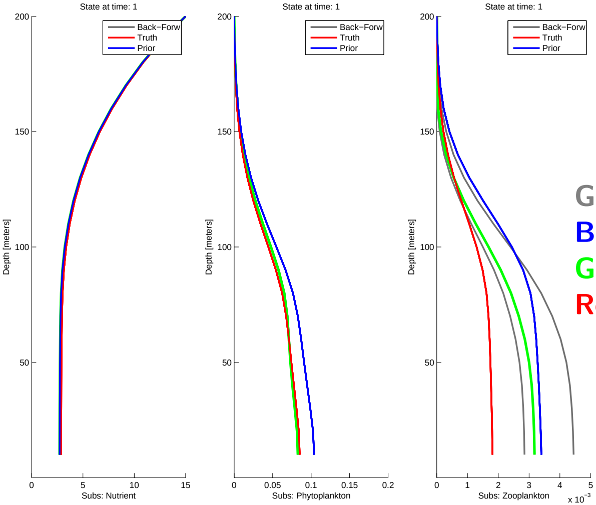
Observation Setup

Observe

What	surf Phyto
When	every 4 days
Error	**1 %**

Experiment 3 - Unrealistic Setup

Initial Condition estimation



Grey - Outer Loops
Blue - Prior
Green - 4DVar
Red - Truth

1. Outline
2. 1D Ecological Model
3. Model Reduced 4D-Var
4. Results
5. **Conclusions**

Conclusions

Model Reduced 4D-Var for 1D Ecosystem Model

- the parameter estimation - works fine when a good quality initial condition is available
- the combined parameter and initial condition estimation - needs an improvement

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Conclusions

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- the parameter estimation - works fine when a good quality initial condition is available
- the combined parameter and initial condition estimation - needs an improvement

- Model Reduced 4D-Var is a potential tool in ecosystem applications

Future Work

- improve the combined parameter and initial condition approach
- apply the method to a large ecosystem model

Thank you for your attention!

Questions?